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# On symmetries of compact Riemann surfaces with cyclic groups of automorphisms

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## Abstract

A Riemann surface  $X$  is said to be of type  $(n, m)$  if its full automorphism group  $\text{Aut } X$  is cyclic of order  $n$  and the quotient surface  $X / \text{Aut } X$  has genus  $m$ . In this paper we determine necessary and sufficient conditions on the integers  $n, m, g$  and  $\gamma$ , where  $n$  is odd, for the existence of a Riemann surface of genus  $g$  and type  $(n, m)$  admitting a symmetry with  $\gamma$  ovals.

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## 1. Introduction

A *symmetry* on a compact Riemann surface  $X$  is an anticonformal involution  $\tau : X \rightarrow X$ . Surfaces admitting some symmetry are called *symmetric*. Under the well-known equivalence between compact Riemann surfaces and (smooth, projective) complex algebraic curves, the symmetric ones correspond to *real algebraic curves*, that is, curves which may be defined over the field  $\mathbb{R}$  of real numbers. The fixed point set of a symmetry consists of  $\gamma$  simple closed

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curves, called *ovals* in Hilbert's terminology. The seminal work of Harnack [6] asserts that  $0 \leq \gamma \leq g + 1$ , where  $g$  is the genus of the surface. Each oval corresponds to a connected component of the set of  $\mathbb{R}$ -rational points of the associated real algebraic curve. More recently, Natanzon has found upper bounds for the sum of the number of ovals of some distinguished symmetries on the surface, see [10,11].

Let  $X$  be a symmetric compact Riemann surface. We denote by  $\text{Aut}^\pm X$  its full group of conformal and anticonformal automorphisms, and by  $\text{Aut } X$  its subgroup of index 2 consisting of the conformal ones. A Riemann surface  $X$  (not necessarily symmetric) is said to be of type  $(n, m)$  if  $\text{Aut } X$  is the cyclic group of order  $n$  and the quotient surface  $X/\text{Aut } X$  has genus  $m$ . For example, if  $m = 0$  then  $X$  is  $n$ -gonal, and if  $n = 2$  then  $X$  is  $m$ -hyperelliptic.

In this paper we determine necessary and sufficient conditions on the integers  $n, m, g$  and  $\gamma$ , where  $n > 1$  is odd,  $m \geq 1$  and  $g \geq 2$ , for the existence of a Riemann surface of genus  $g$  and type  $(n, m)$  admitting a symmetry with  $\gamma$  ovals.

As particular cases of our results, we should mention the results obtained by Bujalance, Costa and Gamboa in [2] and by Nakamura in [9]. The arguments involved in our proofs are of a different nature, and they rely upon recent techniques which allow the counting of ovals of a symmetry in situations which are more cumbersome than those in [2,9].

## 2. Preliminaries

We will extensively use the combinatorial theory of non-euclidean crystallographic groups and symmetric Riemann surfaces, whose main results for the purposes of this paper are briefly stated in this section. For a general account on this theory we refer the reader to Chapters 0–2 in [3].

A non-euclidean crystallographic (NEC) group is a discrete group of isometries of the hyperbolic plane  $U$  with compact quotient space. The *signature* of an NEC group  $\Delta$  is a collection  $\sigma(\Delta)$  of non-negative integers and symbols of the form

$$\sigma(\Delta) = (h; \pm; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{k1}, \dots, n_{ks_k})\}),$$

which determine the algebraic structure of  $\Delta$  and the geometric nature of the canonical projection  $\pi: U \rightarrow U/\Delta$ . The quotient  $U/\Delta$  is a surface of genus  $h$  whose boundary has  $k \geq 0$  connected components, and it is orientable or not according to the sign being “+” or “−,” respectively. The integers  $m_i \geq 2$  are the *proper periods* of  $\Delta$ , and represent the branching over interior points of  $U/\Delta$  under the projection  $\pi$ . The  $k$  brackets  $(n_{i1}, \dots, n_{is_i})$  are the *period cycles* and represent the branching over the  $i$ th boundary component of  $U/\Delta$ . The integers  $n_{ij} \geq 2$  are the *link periods*. The signature also determines the algebraic structure of  $\Delta$ . It has generators

- $x_1, \dots, x_r$  (elliptic isometries);
- $c_{10}, \dots, c_{1s_1}, \dots, c_{k0}, \dots, c_{ks_k}$  (reflections);
- $e_1, \dots, e_k$  (orientation preserving isometries, which will be called *connecting generators*);
- $a_1, b_1, \dots, a_h, b_h$  (hyperbolic translations) if the sign of  $\sigma(\Delta)$  is “+,” or  $d_1, \dots, d_h$  (glide reflections) otherwise;

and defining relations

- $x_i^{m_i} = 1$  for  $i = 1, \dots, r$ ;

- $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$ , for  $i = 1, \dots, k$ , and  $j = 1, \dots, s_i$ ;
- $e_i c_{i0} e_i^{-1} c_{is_i} = 1$ , for  $i = 1, \dots, k$ ;
- $[a_1, b_1] \cdots [a_h, b_h] x_1 \cdots x_r e_1 \cdots e_k = 1$  if the sign “+” occurs, and  $d_1^2 \cdots d_h^2 x_1 \cdots x_r e_1 \cdots e_k = 1$  otherwise.

A set of generators as the above is called a *set of canonical generators*.

The area of (a fundamental domain for)  $\Delta$  is

$$\mu(\Delta) = 2\pi \left( \eta h + k - 2 + \sum_{i=1}^r \left( 1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left( 1 - \frac{1}{n_{ij}} \right) \right),$$

where  $\eta = 2$  if  $\sigma(\Delta)$  has sign “+” and  $\eta = 1$  otherwise. If  $\Lambda$  is a subgroup of  $\Delta$  of finite index, then  $\Lambda$  is also an NEC group and the Riemann–Hurwitz formula reads

$$\mu(\Lambda) = [\Delta : \Lambda] \cdot \mu(\Delta).$$

An NEC group with no orientation reversing isometries is called a *Fuchsian group*, and its signature is abbreviated by  $(h; m_1, \dots, m_r)$ . If such a group contains no non-trivial element of finite order then it is said to be a *surface Fuchsian group*, and its signature is abbreviated by  $(h; -)$ .

A compact Riemann surface  $X$  of genus  $g \geq 2$  can be represented as the orbit space  $U/\Gamma$  for some Fuchsian surface group  $\Gamma$  with signature  $(g; -)$ . Given a surface  $X$  so represented, an abstract finite group  $G$  acts as a group of conformal and anticonformal automorphisms of  $X$  if and only if there exist an NEC group  $\mathcal{N}$  and an epimorphism  $\theta: \mathcal{N} \rightarrow G$  with  $\ker \theta = \Gamma$ . An epimorphism whose kernel is a surface Fuchsian group will be called *smooth epimorphism*.

In particular, the group  $\text{Aut}^\pm X$  of conformal and anticonformal automorphisms of a surface  $X = U/\Gamma$  is isomorphic to the factor group  $N_\Gamma/\Gamma$ , where  $N_\Gamma$  is the normalizer of  $\Gamma$  in the group  $\text{PGL}(2; \mathbb{R})$  of conformal and anticonformal automorphisms of the hyperbolic plane  $U$ . The subgroup  $\text{Aut} X$  of conformal automorphisms of  $X$  is isomorphic to the quotient  $(N_\Gamma)^+/\Gamma$ , where  $(N_\Gamma)^+$  is the *canonical Fuchsian subgroup* of  $N_\Gamma$ , that is, the subgroup of  $N_\Gamma$  consisting of its orientation preserving elements.

### 3. On conjugacy classes of symmetries

Let  $\tau$  be a symmetry on a compact Riemann surface  $X$  of genus  $g \geq 2$  whose full group  $\text{Aut} X$  of conformal automorphisms is cyclic of odd order  $n > 1$ . Let  $u$  be a generator of  $\text{Aut} X$ . Then  $\tau$  normalizes  $\langle u \rangle$ , that is,  $\tau u \tau \in \langle u \rangle$ . So there exists a positive integer  $\alpha < n$  with  $\alpha^2 \equiv 1 \pmod{n}$  such that the full group  $\text{Aut}^\pm X$  of conformal and anticonformal automorphisms of  $X$  is the group  $G_n^\alpha$  with presentation

$$G_n^\alpha := \langle u, \tau \mid u^n = \tau^2 = 1, \tau u \tau = u^\alpha \rangle. \quad (1)$$

For a fixed triple  $(g, n, m)$ , let  $\Sigma_{(g,n,m)}$  be the family of genus  $g$  symmetric Riemann surfaces  $X$  of type  $(n, m)$ . The above groups  $G_n^\alpha$  provide a partition of  $\Sigma_{(g,n,m)}$  into finitely many subfamilies  $\Sigma_{(g,n,m)}^\alpha$  consisting, for each value of  $\alpha$ , of those  $X \in \Sigma_{(g,n,m)}$  with  $\text{Aut}^\pm X = G_n^\alpha$ . For

example,  $\Sigma_{(g,n,m)}^1$  consists of those surfaces  $X$  whose full group  $\text{Aut}^\pm X = G_n^1$  is the cyclic group of order  $2n$ , while those in  $\Sigma_{(g,n,m)}^{n-1}$  have dihedral full automorphism group.

The elements of  $G_n^\alpha$  can be written as  $u^k \tau^i$  for some  $k \in \{0, \dots, n-1\}$  and  $i \in \{0, 1\}$ . Those involutions in  $G_n^\alpha$  with  $i = 1$  are the symmetries of  $X$ , and so we call them the *symmetries* in  $G_n^\alpha$ .

The following notations will appear throughout the paper. We set

$$d = \gcd(n, \alpha + 1) \quad \text{and} \quad f = \gcd(n, \alpha - 1),$$

where  $\gcd$  stands for the greatest common divisor. Note that  $\gcd(d, f) = 1$  since  $\gcd(\alpha + 1, \alpha - 1) = 1$  or  $2$  and  $n$  is odd. Hence, the product  $d \cdot f$  divides  $n$ . Moreover, we claim that

$$d \cdot f = n. \quad (2)$$

Indeed, if  $p^k$  is a maximal prime power dividing  $n$  then it also divides  $\alpha^2 - 1 = (\alpha + 1)(\alpha - 1)$ . Hence,  $p^k$  divides either  $\alpha + 1$  or  $\alpha - 1$ , and so  $p^k$  divides either  $d$  or  $f$ . This proves equality (2).

We are ready to prove a purely group theoretic lemma concerning the conjugacy classes of symmetries in  $G_n^\alpha = \text{Aut}^\pm X$ .

**Lemma 3.1.** *In the conditions above, the Riemann surface  $X$  has a unique conjugacy class of symmetries in  $\text{Aut}^\pm X$ .*

**Proof.** If  $u^i \tau$  is a symmetry then after squaring, we get  $u^{i(\alpha+1)} = 1$ , that is  $i(\alpha + 1) \equiv 0 \pmod{n}$ . Thus  $f = n/d$  divides  $i(\alpha + 1)/d$  and, since  $f$  and  $\alpha + 1$  are coprime, we deduce that the symmetries in  $G_n^\alpha$  are the elements of the form  $u^{hf} \tau$  for  $h = 1, \dots, d$ . For each such value of  $h$ , there exists an integer  $y_h$  such that  $(1 - \alpha)y_h \equiv hf \pmod{n}$  (by the very definition of  $f$ ). It is straightforward to check that this integer makes the equality

$$u^{-y_h} (u^{hf} \tau) u^{y_h} = \tau$$

to hold. This means that all symmetries in  $G_n^\alpha$  are conjugate to  $\tau$ .  $\square$

**Remarks 3.2.** (1) It follows from the equality  $df = n$ , that

$$f(\alpha + 1) \equiv d(\alpha - 1) \equiv 0 \pmod{n}.$$

Note also that  $\gcd(f, \alpha + 1) = \gcd(d, \alpha - 1) = 1$ . We will use these equalities later on.

(2) Also for later purposes, we remark that the symmetries of the surface  $X$  are the elements of the form  $u^{hf} \tau$  with  $h \in \{1, \dots, d\}$ .

(3) We are interested in computing the number of ovals of symmetries of  $X$ . Since all of them are pairwise conjugate, we just have to deal with one of them, for example, with the symmetry  $\tau$  we started with.

In order to compute the number  $\|\tau\|$  of ovals of the symmetry  $\tau$ , we will need a description of the centralizer of a reflection  $c$  in an NEC group  $\Delta$ . Singerman [13] proved that if  $c$  corresponds to an empty period cycle or to a period cycle all of whose link periods are odd, then its centralizer is isomorphic to  $C_2 \times C_\infty$ , where  $C_n$  denotes the cyclic group of order  $n$ , and  $C_\infty$  the cyclic group of infinite order. Going a bit more into the details of Singerman's proof, one can find

explicitly the generators of this group. Indeed, we have the following lemma, whose proof is similar to [5, Theorem 3] and we omit it.

**Lemma 3.3.** *Let  $e, c_0, \dots, c_s$  be a set of canonical generators associated to a period cycle  $(n_1, \dots, n_s)$ , with  $n_i$  odd for all  $i$ , of an NEC group  $\Delta$ , and let  $C(\Delta, c_0)$  be the centralizer in  $\Delta$  of  $c_0$ .*

- (i) *If  $s = 0$  then  $C(\Delta, c_0) = \langle c_0 \rangle \oplus \langle e \rangle$ .*
- (ii) *If  $s \neq 0$  then  $C(\Delta, c_0) = \langle c_0 \rangle \oplus \langle \prod_{i=0}^{s-1} (c_{i+1} c_i)^{(n_{i+1}-1)/2} \cdot e \rangle$ .*

#### 4. On the number of ovals of a symmetry

Let  $\tau$  be a representative of the unique conjugacy class of symmetries on a symmetric Riemann surface  $X$  of type  $(n, m)$  with  $n$  odd. Let us write  $X = U/\Gamma$  for some surface Fuchsian group  $\Gamma$ , and  $G_n^\alpha = \text{Aut}^\pm X = N_\Gamma/\Gamma$ , where  $N_\Gamma$  is the normalizer of  $\Gamma$  in  $\text{PGL}(2; \mathbb{R})$ . We denote  $f = \gcd(n, \alpha - 1)$ . Finally let  $\theta : N_\Gamma \rightarrow G_n^\alpha$  denote the corresponding smooth epimorphism. With these notations we have the following theorem.

**Theorem 4.1.** *Let  $e_1, \dots, e_k$  be a set of connecting generators associated to the  $k$  period cycles of the signature of  $N_\Gamma$ . Then  $\theta(e_i) = u^{\varepsilon_i}$  for some  $\varepsilon_i \in \{1, \dots, n\}$ , and the number of ovals of  $\tau$  is*

$$\|\tau\| = \sum_{i=1}^k \gcd(\varepsilon_i, f).$$

*If there is no period cycle in the signature of  $N_\Gamma$ , then  $\|\tau\| = 0$ .*

**Proof.** Observe first that  $\theta(e_i)$  is indeed of the form  $u^{\varepsilon_i}$  since  $\theta(e_i)$  belongs to the orientation preserving subgroup  $\text{Aut } X = \langle u \rangle$  of  $\text{Aut}^\pm X$ . If there is no period cycle then  $N_\Gamma$  contains no reflection and  $\tau$  has to be the image of a glide reflection. It follows that the fixed point set of  $\tau$  is empty, that is,  $\|\tau\| = 0$ . If  $k \geq 1$  then  $\tau = \theta(c)$  for some reflection  $c \in N_\Gamma$  and so the fixed point set of  $\tau$  is non-empty. In this case, Theorem 3.1 in [4] shows that the number  $\|\tau\|$  of ovals of  $\tau$  is

$$\sum_i [C(G_n^\alpha, \theta(c_i)) : \theta(C(N_\Gamma, c_i))], \quad (3)$$

where  $C(G, g)$  denotes the centralizer in the group  $G$  of the element  $g \in G$ , and the sum is taken over a set of representatives of conjugacy classes of canonical reflections in  $N_\Gamma$  whose images are conjugate to  $\tau$ . In our case, all the link periods of the  $k$  period cycles of  $N_\Gamma$  are odd and so all the reflections associated to the same period cycle are conjugate to each other. In addition, by Lemma 3.1, all the images in  $G_n^\alpha$  of the canonical reflections are conjugate to  $\tau$  and therefore the sum (3) has exactly  $k$  terms, one for each period cycle of  $N_\Gamma$ . Our goal is to show that each term equals  $\gcd(\varepsilon_i, f)$ . In all of them we have  $|C(G_n^\alpha, \theta(c_i))| = |G_n^\alpha|/d = 2f$  since the number of conjugates of  $\theta(c_i)$  in  $G_n^\alpha$  is  $d$  for all  $c_i$ , see (2) in Remarks 3.2. Here  $|\cdot|$  denotes the order of a group.

Let us fix a period cycle of  $N_\Gamma$ , denote by  $e$  its connecting generator and write  $\theta(e) = u^\varepsilon$ .

Suppose first that the period cycle is empty, and let  $c_0$  be a canonical reflection associated to it. Lemma 3.3 shows that  $|\theta(C(N_\Gamma, c_0))| = 2 \cdot |\langle u^\varepsilon \rangle| = 2n/\gcd(\varepsilon, n)$  and so the contribution

to  $\|\tau\|$  of an empty period cycle is  $\gcd(\varepsilon, n)f/n = \gcd(\varepsilon, n)/d$ . It follows from the relation  $ec_0 = c_0e$  that  $\varepsilon(\alpha - 1) \equiv 0 \pmod{n}$  and hence  $\varepsilon \equiv 0 \pmod{d}$ , since  $(\alpha - 1)/f$  and  $d$  are coprime. Therefore  $\gcd(\varepsilon, n)/d = \gcd(\varepsilon/d, f) = \gcd(\varepsilon, f)$  where in the last equality we have used that  $d$  and  $f$  are coprime.

Let us consider now a non-empty period cycle  $(n_1, \dots, n_s)$  of  $N_\Gamma$ , and let  $c_0, \dots, c_s$  be a set of canonical reflections associated to it. The image  $\theta(c_i)$  of each reflection  $c_i$  is a symmetry in  $G_n^\alpha = \langle u, \tau \rangle$  and so it has the form  $\theta(c_i) = u^{h_i f} \tau$  for some  $h_i \in \{1, \dots, d\}$ , see (2) in Remarks 3.2. In particular,

$$\theta(c_{i+1}c_i) = u^{f(h_{i+1} + \alpha h_i)} = u^{f(h_{i+1} - h_i)},$$

where in the last equality we have used that  $f\alpha \equiv -f \pmod{n}$ , see (1) in Remarks 3.2. Lemma 3.3 shows that the order of  $\theta(C(N_\Gamma, c_0))$  is twice the order of the element  $a := \prod_{i=0}^{s-1} \theta(c_{i+1}c_i)^{(n_{i+1}-1)/2} \cdot \theta(e)$ . Since all the factors  $\theta(c_{i+1}c_i)$  and  $\theta(e)$  belong to the odd order cyclic group  $\langle u \rangle$ , then the order of  $a$  coincides with the order of its square. Now,

$$\theta(c_{i+1}c_i)^{n_{i+1}-1} = \theta(c_{i+1}c_i)^{-1} = u^{f(h_i - h_{i+1})}$$

and so  $a^2 = u^t$  with

$$t = \sum_{i=0}^{s-1} f(h_i - h_{i+1}) + 2\varepsilon = f(h_0 - h_s) + 2\varepsilon.$$

Now, the relation  $ec_0 = c_s e$  gives  $f(h_0 - h_s) \equiv \varepsilon(\alpha - 1) \pmod{n}$ , and so  $t \equiv \varepsilon(\alpha + 1) \pmod{n}$ . Therefore

$$|\theta(C(N_\Gamma, c_0))| = \frac{2n}{\gcd(\varepsilon(\alpha + 1), n)} = \frac{2f}{\gcd(\varepsilon(\alpha + 1)/d, f)} = \frac{2f}{\gcd(\varepsilon, f)}.$$

So the contribution of a non-empty period cycle to  $\|\tau\|$  is also  $\gcd(\varepsilon, f)$ .  $\square$

## 5. Symmetries with prescribed number of ovals

In this section we determine necessary and sufficient conditions on the integers  $n, m, g$  and  $\gamma$ , where  $n > 1$  is odd and  $m > 0$ , for the existence of a symmetric Riemann surface of genus  $g \geq 2$  and type  $(n, m)$  whose unique conjugacy class of symmetries has  $\gamma$  ovals.

First we introduce some terminology. We say that the family of positive integers  $\{m_1, \dots, m_t\}$  satisfies the *elimination property* if

$$\text{lcm}\{m_1, \dots, \hat{m}_i, \dots, m_t\} = \text{lcm}\{m_1, \dots, m_t\} \quad \text{for each } i = 1, \dots, t,$$

where  $\hat{m}_i$  denotes the omission of  $m_i$  and lcm stands for the least common multiple. We adopt the convention that lcm of the empty set is 1. Thus, the set  $\{m_1\}$  has the elimination property if and only if  $m_1 = 1$ . The following lemma will be used in the proofs of Theorems 5.2 and 5.4. We provide a proof of this elementary lemma for the reader's convenience. A more general result is proved in [3, Lemma 3.1.1]. We denote by  $\mathbb{Z}_n$  the additive group of integers mod  $n$ , and by  $[x]_n$  the class mod  $n$  of  $x \in \mathbb{Z}$ .

**Lemma 5.1.** *Let  $m_1, \dots, m_t$  be odd integers and  $M = \text{lcm}\{m_1, \dots, m_t\}$ . The following statements are equivalent:*

- (1) *The integers  $\{m_1, \dots, m_t\}$  satisfy the elimination property.*
- (2) *For each multiple  $M'$  of  $M$  there exist classes  $\mu_1, \dots, \mu_t \in \mathbb{Z}_{M'}$  such that  $\text{ord}(\mu_i) = m_i$  and  $\mu_1 + \dots + \mu_t = 0$  in  $\mathbb{Z}_{M'}$ .*

**Proof.** (1)  $\Rightarrow$  (2). There exists an injective group homomorphism  $\mathbb{Z}_M \rightarrow \mathbb{Z}_{M'}$  given by  $[x]_M \mapsto [dx]_{M'}$  where  $dM = M'$ , and so it is enough to consider the case  $M' = M$ . We first deal with the case  $M = p^k$  where  $p$  is an odd prime. We will prove the existence of integers  $a_1, \dots, a_t$  such that

$$\gcd(a_i, m_i) = 1 \quad \text{and} \quad \sum_{i=1}^t a_i M/m_i \text{ is a multiple of } M.$$

Then the classes  $\mu_i = [a_i M/m_i]_M$  satisfy the required conditions. The hypothesis means, in this case, that  $m_i = M = p^k$  for at least two indices  $i$ . So we can suppose that each  $m_i = p^{l_i}$  with  $l_i = k$  precisely for  $t - s + 1 \leq i \leq t$  and some  $2 \leq s \leq t$ . Note that the quotients  $M/m_i = p^{k-l_i}$  are multiple of  $p$  for  $1 \leq i \leq t - s$ , and so there exists an integer  $R$  such that  $\sum_{i=1}^{t-s} M/m_i = Rp$ . Let us distinguish now two subcases, according to  $s \equiv 1 \pmod{p}$  or not. In the first case an straightforward computation shows that the choice

$$a_i = 1 \quad \text{for } 1 \leq i \leq t - 2; \quad a_{t-1} = -s \quad \text{and} \quad a_t = 2 - Rp$$

works, whilst if  $s \not\equiv 1 \pmod{p}$  we choose

$$a_i = 1 \quad \text{for } 1 \leq i \leq t - 1 \quad \text{and} \quad a_t = 1 - s - Rp.$$

In the general case  $M = \prod_{j=1}^r p_j^{k_j}$ , where each  $p_j$  is prime, we have  $m_i = \prod_{j=1}^r p_j^{l_{ij}}$  for certain non-negative integers  $l_{ij}$ . It is easily seen that for each index  $j$  the family  $\{m_{ij} = p_j^{l_{ij}} : 1 \leq i \leq t\}$  also satisfies the elimination property. Thus there exists a set of integers  $\{a_{ij} : 1 \leq i \leq t, 1 \leq j \leq r\}$  such that

$$\gcd(a_{ij}, m_{ij}) = 1 \quad \text{and} \quad \sum_{i=1}^t a_{ij} p_j^{k_j - l_{ij}} \text{ is a multiple of } p_j^{k_j} \quad \text{for each index } j.$$

Since the integers  $\{p_1^{k_1}, \dots, p_r^{k_r}\}$  are coprime we apply the Chinese Remainder Theorem to deduce the existence of integers  $\{\eta_i : 1 \leq i \leq t\}$  such that

$$\eta_i \equiv a_{ij} p_j^{k_j - l_{ij}} \pmod{p_j^{k_j}} \quad \text{for each } 1 \leq j \leq r.$$

Then the classes  $\{\mu_i = [\eta_i]_M : 1 \leq i \leq t\}$  satisfy the statement. Indeed, since

$$\text{ord}[\eta_i]_{p_j^{k_j}} = \text{ord}[a_{ij} p_j^{k_j - l_{ij}}]_{p_j^{k_j}} = \text{ord}[p_j^{k_j - l_{ij}}]_{p_j^{k_j}} = p_j^{l_{ij}} = m_{ij},$$

we get that for  $1 \leq i \leq t$ ,

$$\text{ord}(\mu_i) = \text{lcm}\{m_{ij}: 1 \leq j \leq r\} = m_i.$$

Moreover, for each  $1 \leq j \leq r$ ,

$$\sum_{i=1}^t \mu_i \equiv \sum_{i=1}^t a_{ij} p_j^{k_j - l_{ij}} \equiv 0 \pmod{p_j^{k_j}}$$

and so  $\sum_{i=1}^t \mu_i$  is a multiple of  $M$ .

(2)  $\Rightarrow$  (1). We choose  $M' = M$  and denote by  $H_i$  the subgroup of  $\mathbb{Z}_M$  generated by the classes  $\mu_1, \dots, \mu_{i-1}, \mu_{i+1}, \dots, \mu_t$ . By assumption  $\mu_i \in H_i$ , and so its order  $m_i$  divides the order of  $H_i$ . But

$$\text{ord}(H_i) = \text{lcm}\{\text{ord}(\mu_k) = m_k: k \neq i\},$$

and so the integers  $\{m_1, \dots, m_t\}$  satisfy the elimination property.  $\square$

Having disposed of this preliminary step, we can now return to deal with symmetric Riemann surfaces. We first consider the case of surfaces  $X$  of type  $(n, m)$  such that the quotient  $X/\text{Aut}^\pm X$  is orientable. Observe that this implies that such quotient has non-empty boundary, and so the number  $\gamma$  of ovals of the symmetry  $\tau$  of  $X$  is positive.

**Theorem 5.2.** *Let  $g, n, m, \gamma$  and  $\alpha$  be positive integers with  $g \geq 2$ ,  $n > 1$  odd and  $\alpha^2 \equiv 1 \pmod{n}$ . Then there exists a genus  $g$  symmetric Riemann surface  $X$  of type  $(n, m)$  with  $\text{Aut}^\pm X = G_n^\alpha$  and orientable quotient  $X/\text{Aut}^\pm X$ , whose unique symmetry (up to conjugacy within  $\text{Aut}^\pm X$ ) has  $\gamma$  ovals if and only if there exist*

- divisors  $m_1, \dots, m_r$  of  $n$ , with  $m_i > 1$  for all  $i$ ,
- divisors  $n_{11}, \dots, n_{1s_1}, \dots, n_{t1}, \dots, n_{ts_t}$  of  $d = \gcd(n, \alpha + 1)$ , with  $n_{ij} > 1$  for all  $i, j$ ,
- non-negative integers  $p_{11}, \dots, p_{1s_1}, \dots, p_{t1}, \dots, p_{ts_t}$  with  $\gcd(p_{ij}, n_{ij}) = 1$  for all  $i, j$ ,
- non-negative integers  $\varepsilon_1, \dots, \varepsilon_k$  with  $k \geq 1$  and  $k \geq t$ ,

such that

- (1)  $m + 1 - k$  is a non-negative even integer;
- (2)  $\frac{g-1}{n} = m - 1 + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{i=1}^t \sum_{j=1}^{s_i} (1 - \frac{1}{n_{ij}})$ ;
- (3)  $2m + 2r + \sum_{i=1}^t s_i > 4$ ;
- (4) the set  $\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  satisfies the elimination property, where  $\varepsilon'_i = n / \gcd(n, \varepsilon_i)$ ;
- (5) if  $k = m + 1$  then  $\text{lcm}\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  is a multiple of  $f = \gcd(n, \alpha - 1)$ ;
- (6)  $\varepsilon_i \cdot \frac{\alpha-1}{f} \equiv \sum_{j=1}^{s_i} \frac{d}{n_{ij}} \cdot p_{ij} \pmod{d}$  for each  $i = 1, \dots, t$ ;
- (7)  $\varepsilon_{t+i} \equiv 0 \pmod{d}$  for  $i = 1, \dots, k - t$ ;
- (8)  $\gamma = \sum_{i=1}^k \gcd(\varepsilon_i, f)$ .

**Proof.** Let us assume first that there exists a symmetric Riemann surface  $X$  satisfying the hypothesis of the theorem. Then  $X = U/\Gamma$  for some surface Fuchsian group  $\Gamma$  with signature  $\sigma(\Gamma) = (g; -)$ . The normalizer of  $\Gamma$  in  $\text{PGL}(2; \mathbb{R})$  is a proper NEC group  $N_\Gamma$  so that



$\text{Aut}^\pm X = N_\Gamma / \Gamma$ . Observe that the sign in the signature of  $N_\Gamma$  is “+” because the quotient surface  $X / \text{Aut}^\pm X = U / N_\Gamma$  is orientable. Let  $g'$  and  $k$  be the genus and the number of boundary components of  $X / \text{Aut}^\pm X$ , respectively; then  $2g' + k - 1 = m$  because  $m$  is the genus of its orientable double covering  $X / \text{Aut} X$ . So  $m + 1 - k$  is even and non-negative, which proves assertion (1), and the signature of  $N_\Gamma$  has the form

$$((m + 1 - k)/2; +; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{t1}, \dots, n_{ts_t}), (-)^{k-t}\}),$$

for some  $m_i > 1$  and  $n_{ij} > 1$  which, as we shall see, satisfy the conditions in the statement. Observe that  $k \geq 1$  since  $N_\Gamma$  has to contain some reflection because  $\gamma > 0$ .

Let  $\theta: N_\Gamma \rightarrow G_n^\alpha = \langle u, \tau \rangle$  be the corresponding smooth epimorphism with  $\ker \theta = \Gamma$ . The images  $\theta(x_i)$  ( $i = 1, \dots, r$ ) of the elliptic canonical generators of  $N_\Gamma$  lie in the orientation preserving subgroup  $\langle u \rangle$  of  $G_n^\alpha$ , and so their orders  $m_i$  are divisors of  $n = |\langle u \rangle|$ . The images  $\theta(c_{ij})$  of the canonical reflections in  $N_\Gamma$  are symmetries in  $G_n^\alpha$  and so they have the form  $u^{f h_{ij}} \tau$ . Since  $\theta$  preserves orders of elliptic elements, and  $n_{ij} = \text{ord}(c_{ij-1} c_{ij})$ , then also  $n_{ij} = \text{ord}(\theta(c_{ij-1}) \theta(c_{ij})) = \text{ord}(u^{f(h_{ij-1} + \alpha h_{ij})})$ , which divides  $d$ .

Assertion (2) is the Riemann–Hurwitz formula applied to  $\Gamma$  and  $N_\Gamma$ . Assertion (3) follows from the fact that the group  $\text{Aut} X$  is cyclic. To show this we use Theorem 4.1 in [1], where the extendability of cyclic actions is carefully studied. First,  $\text{Aut} X = (N_\Gamma)^+ / \Gamma$  where  $(N_\Gamma)^+$  is the canonical Fuchsian group of  $N_\Gamma$ , whose signature is

$$\sigma((N_\Gamma)^+) = (m; m_1, m_1, \dots, m_r, m_r, n_{11}, \dots, n_{1s_1}, \dots, n_{t1}, \dots, n_{ts_t}),$$

see [3, Corollary 2.2.5]. Assume  $m = 1$ ; then  $2r + \sum s_i > 0$  for  $(N_\Gamma)^+$  to have positive area; the values  $r = 0$  and  $\sum s_i = 1$  cannot occur since there is no smooth epimorphism from a Fuchsian group with signature  $(1; n_1)$  onto  $C_n$ ; if  $r = 0$  and  $\sum s_i = 2$  then these two periods have to coincide by Theorem 4.(i) in [7]; so  $\sigma((N_\Gamma)^+)$  has the same signature as in the case  $r = 1$ ,  $\sum s_i = 0$ ; in both cases Theorem 4.1 in [1] assures that the action of  $C_n$  on  $X$  would extend to a larger group. The same happens if  $m = 2$  and  $r = \sum s_i = 0$ . This way assertion (3) is proved.

We now prove assertion (4). With the same notations as in Theorem 4.1, we write  $\theta(e_i) = u^{\varepsilon_i}$  for  $i = 1, \dots, k$ , where  $e_i$  is the connecting generator of the  $i$ th period cycle of the signature of  $N_\Gamma$ . Thus  $\varepsilon'_i := n / \gcd(n, \varepsilon_i)$  is the order of  $\theta(e_i)$ . The canonical relation  $\prod_{i=1}^{g'} [a_i, b_i] \prod_{i=1}^r x_i \prod_{i=1}^k e_i = 1$  in  $N_\Gamma$  involves only orientation preserving elements and so it implies  $\prod_{i=1}^r \theta(x_i) \prod_{i=1}^k \theta(e_i) = 1$ , a relation in  $C_n$ . In fact, this is a relation in the subgroup  $\langle \theta(x_i), \theta(e_j) \rangle \cong C_M$  where  $M = \text{lcm}\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$ . So it is also a relation in  $C_{M'}$  for each multiple  $M'$  of  $M$ . It follows that there exist  $\mu_1, \dots, \mu_r, \xi_1, \dots, \xi_k$  in  $\mathbb{Z}_{M'}$  such that  $\text{ord}(\mu_i) = \text{ord}(\theta(x_i)) = m_i$ ,  $\text{ord}(\xi_i) = \text{ord}(\theta(e_i)) = \varepsilon'_i$  and  $\sum_{i=1}^r \mu_i + \sum_{i=1}^k \xi_i = 0$  in  $\mathbb{Z}_{M'}$ . Lemma 5.1 yields then that  $\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  satisfies the elimination property, which proves assertion (4).

Let us check assertion (5). If  $k = m + 1$  then  $\text{im } \theta = G_n^\alpha$  is generated by  $\theta(x_i)$  ( $i = 1, \dots, r$ ),  $\theta(e_j)$  ( $j = 1, \dots, k$ ), and the images  $\theta(c_{ij})$  of the canonical reflections in  $N_\Gamma$ . These last generate a group of the form  $\langle \tau, u^{hf} \rangle$  for some  $h \geq 0$ , while  $\langle \theta(x_i), \theta(e_j) \rangle = \langle u^{n/M} \rangle$  with  $M$  as above. Therefore  $\langle u \rangle = \langle u^{hf}, u^{n/M} \rangle$  and so  $\gcd(f, n/M) = 1$ . Since  $f$  divides  $n$ , it follows that  $M$  is a multiple of  $f$ , which is assertion (5).

Assertion (6) deals with the  $t$  non-empty period cycles of the signature of  $N_\Gamma$ . To lighten notation, let  $(n_1, \dots, n_s)$  be one of them and let  $c_0, \dots, c_s, e$  be the set of canonical generators associated to it. We want so show assertion (6) with the subscript  $i$  deleted. Recall that  $\theta(c_j) =$

$u^{h_j f} \tau$  for some  $h_j \in \{0, \dots, d\}$  and so  $\theta(c_{j-1}c_j) = u^{f(h_{j-1}+\alpha h_j)} = u^{f(h_{j-1}-h_j)}$ . Since  $n_j$  is the order of this element, we get

$$n_j = \frac{n}{\gcd(f(h_{j-1}-h_j), n)} = \frac{d}{\gcd(h_{j-1}-h_j, d)}$$

and so there exists an integer  $p_j$  prime with  $n_j$  such that  $h_{j-1}-h_j = p_j(d/n_j)$ . Therefore  $h_0-h_s = \sum_{j=1}^s p_j(d/n_j)$ . Substituting this equality in  $\varepsilon(\alpha-1) \equiv f(h_0-h_s) \pmod{n}$  (which comes from the relation  $ec_0 = c_s e$ ), we get assertion (6).

The last  $k-t$  period cycles are empty and so the connecting generator  $e_{t+i}$  and the reflection  $c_{t+i,0}$  commute ( $1 \leq i \leq k-t$ ). It follows that  $\varepsilon_{t+i}(\alpha-1) \equiv 0 \pmod{n}$  and so  $\varepsilon_{t+i} \equiv 0 \pmod{d}$ . This proves assertion (7).

Assertion (8) is the formula in Theorem 4.1.

Conversely, let  $g, n, m, \gamma, \alpha, t, k, m_s, n_{ij}, p_{ij}, \varepsilon_j$  be integers satisfying the conditions in the statement of the theorem. We are going to construct a genus  $g$  symmetric Riemann surface  $X$  of type  $(n, m)$  whose full group  $\text{Aut}^\pm X$  is isomorphic to the abstract finite group  $G_n^\alpha$  with presentation (1), and orientable quotient  $X/\text{Aut}^\pm X$ , whose unique symmetry (up to conjugacy within  $\text{Aut}^\pm X$ ) has  $\gamma$  ovals.

Let us consider the signature

$$\sigma = (g'; +; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{t1}, \dots, n_{ts_t}), (-)^{k-t}\}), \quad (4)$$

where  $g' = (m+1-k)/2$ . There exist NEC groups with signature  $\sigma$  since its area is positive as is easy to check using condition (3). In addition, the same condition (3) assures that the signature  $\sigma^+$  of the canonical Fuchsian groups of such NEC groups does not occur in Singerman's list [12] of non-maximal Fuchsian signatures. Therefore  $\sigma$  is a maximal NEC signature and we may choose a maximal NEC group  $\mathcal{N}$  (that is,  $\mathcal{N}$  is not properly contained in another NEC group) with  $\sigma(\mathcal{N}) = \sigma$  (see Section 5.1 in [3]). We want to define a smooth epimorphism  $\theta: \mathcal{N} \rightarrow G_n^\alpha$  in such a way that  $X = U/\ker \theta$  is the surface we are looking for. To that end we need some preparation.

Recall that we defined  $\varepsilon'_i$  to be  $n/\gcd(n, \varepsilon_i)$ . Let us consider the integer  $M = \text{lcm}\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$ , which is a divisor of  $n$ . Since  $\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  satisfies the elimination property, Lemma 5.1 assures the existence of elements  $\mu_1, \dots, \mu_r, \xi_1, \dots, \xi_k$  in  $\mathbb{Z}_n$  such that

$$\text{ord}(\mu_i) = m_i, \quad \text{ord}(\xi_i) = \varepsilon'_i \quad \text{and} \quad \sum_{i=1}^r \mu_i + \sum_{i=1}^k \xi_i = 0 \quad \text{in } \mathbb{Z}_n.$$

The class of  $\varepsilon_i \pmod{n}$ , has the same order  $\varepsilon'_i$  as  $\xi_i$ . Thus, for each  $i = 1, \dots, k$ , we may choose an integer  $\eta_i$  such that

$$\eta_i \text{ is prime with } n \quad \text{and} \quad \xi_i \equiv \varepsilon_i \cdot \eta_i \pmod{n}.$$

Consider now, for  $1 \leq j \leq s_i$ , the integers

$$h_{ij} = \begin{cases} -\eta_i \sum_{l=1}^j p_{il}(d/n_{il}) & \text{if } k \neq i \leq t; \\ \eta_i(1 - \sum_{l=1}^j p_{il}(d/n_{il})) & \text{if } i = k = t. \end{cases}$$

Let  $\{a_l, b_l, x_s, e_i, c_{ij}\}$  be a set of canonical generators for the NEC group  $\mathcal{N}$  and let us consider the epimorphism  $\theta: \mathcal{N} \rightarrow G_n^\alpha = \langle u, \tau \rangle$  induced by the assignment

- $\theta(a_l) = \theta(b_l) = u, 1 \leq l \leq g'$  (if  $g' \geq 1$ ),
- $\theta(x_s) = u^{\mu_s}, 1 \leq s \leq r$  (if  $r \geq 1$ ),
- $\theta(e_i) = u^{\varepsilon_i \eta_i}, 1 \leq i \leq k$ ,
- $\theta(c_{i0}) = \tau, 1 \leq i \leq k-1, \theta(c_{k0}) = u^{f \eta_k} \tau$ ,
- $\theta(c_{ij}) = u^{f h_{ij}} \tau$  for  $1 \leq j \leq s_i$ .

Using conditions (6) and (7) it is straightforward to see that  $\theta$  preserves the relations in  $\mathcal{N}$  and the finite orders of its elements, that is,  $\theta$  is a well-defined surface kernel homomorphism. We want to prove that it is surjective. This is immediate if  $g' > 0$ . Let us assume now that  $g' = 0$ , i.e.,  $k = m + 1$ . In this case,  $k \geq 2$  and so  $\tau$  and  $u^{f \eta_k} \tau$  belong to  $\text{im } \theta$ ; also  $u^{n/M}$  with  $M = \text{lcm}\{m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  belongs to  $\text{im } \theta$  since  $M$  is the order of the subgroup  $\langle \theta(x_i), \theta(e_j) \rangle$  of  $\langle u \rangle$ . Hence,  $\text{im } \theta$  contains  $\tau$  and the subgroup generated by  $u^{f \eta_k}$  and  $u^{n/M}$ , this last being a group of order  $n$  by condition (5). This shows the surjectivity of  $\theta$ .

Observe that  $\Gamma = \ker \theta$  contains no non-orientable element since these are words with an odd number of canonical reflections  $c_{ij}$  and they are mapped by  $\theta$  onto words with an odd number of letters  $\tau$ . Therefore,  $\Gamma$  is a surface Fuchsian group and we claim that the Riemann surface  $U/\Gamma$  satisfies the requirements of the theorem.

First, the Riemann–Hurwitz formula together with condition (2) yield that this surface has the prescribed genus  $g$ . Since the NEC group  $\mathcal{N}$  has been chosen to be maximal, the full automorphism group  $\text{Aut}^\pm X$  is the quotient  $\mathcal{N}/\Gamma = \langle u, \tau \rangle$ . In particular, the full subgroup  $\text{Aut } X$  of its conformal automorphisms is  $\langle u \rangle = C_n$ , and the quotient space  $X/C_n$  has genus  $2g' + k - 1 = m$ . Hence,  $U/\Gamma$  is a genus  $g$  Riemann surface of type  $(n, m)$  with  $\text{Aut}^\pm X = G_n^\alpha$ . Since  $\text{sign } \sigma(\mathcal{N}) = “+,”$  the quotient  $X/\text{Aut}^\pm X$  is orientable and so it just remains to show that the number  $\gamma$  of ovals of the representative  $\tau$  of the unique conjugacy class of symmetries of  $X$  is  $\gamma = \sum_{i=1}^k \gcd(\varepsilon_i, f)$ . But Theorem 4.1 yields  $\gamma = \sum_{i=1}^k \gcd(\varepsilon_i \eta_i, f) = \sum_{i=1}^k \gcd(\varepsilon_i, f)$ , where the last equality is a consequence of  $\gcd(\eta_i, n) = 1$ .  $\square$

**Remarks 5.3.** (1) The previous Theorem 5.2 extends the results of Nakamura in [9], which deals with the case  $\alpha = 1$ , that is, surfaces  $X = U/\Gamma$  whose full automorphism group is  $\text{Aut}^\pm X = G_n^1 = C_{2n}$ , and with orientable quotient  $X/\text{Aut}^\pm X$ . We point out that this last condition is a consequence of the abelianity of  $\text{Aut}^\pm X$  and [8, Corollary 2].

Another consequence of the abelianity of  $\text{Aut}^\pm X$  is that it simplifies the counting of the number  $\gamma$  of ovals of  $\tau$ , because  $\langle \tau \rangle$  is a normal subgroup in  $\text{Aut}^\pm X$  in this case, which is advantageous, see Chapter 2 in [3].

Moreover, for  $\alpha = 1$ , the group  $G_n^\alpha$  contains a unique element of order 2, and this makes the signature (4) of  $\mathcal{N}$  much simpler. For example, all its period cycles are empty, that is, there is no  $n_{ij}$ .

(2) For a fixed triple  $(g, n, m)$ , the family  $\Sigma_{(g,n,m)}$  of genus  $g$  symmetric Riemann surfaces  $X$  of type  $(n, m)$  splits into finitely many subfamilies  $\Sigma_{(g,n,m)}^\alpha$  consisting of those  $X \in \Sigma_{(g,n,m)}$  with  $\text{Aut}^\pm X = G_n^\alpha$  for a fixed value of  $\alpha$ . Then, in order to get necessary and sufficient conditions for the existence of a Riemann surface  $X \in \Sigma_{(g,n,m)}$  with a symmetry with  $\gamma$  ovals, it suffices to run over the finite number of values of the parameter  $\alpha$ .

We are led now to deal with the case of non-orientable quotient space  $X/\text{Aut}^\pm X$ . Unlike the preceding case, there exist surfaces whose symmetries have no fixed points, that is, the number of ovals of these symmetries is  $\gamma = 0$ . We find it convenient to treat separately the cases  $\gamma = 0$  and  $\gamma > 0$ . We begin with the case in which the symmetry  $\tau$  fixes points; in this case, as in the preceding Theorem 5.2, the NEC group  $\mathcal{N}$  has to contain some reflection.

**Theorem 5.4.** *Let  $g, n, m, \gamma$  and  $\alpha$  be positive integers with  $g \geq 2$ ,  $n > 1$  odd and  $\alpha^2 \equiv 1 \pmod{n}$ . Then there exists a genus  $g$  symmetric Riemann surface  $X$  of type  $(n, m)$  with  $\text{Aut}^\pm X = G_n^\alpha$  and non-orientable quotient  $X/\text{Aut}^\pm X$ , whose unique symmetry (up to conjugacy within  $\text{Aut}^\pm X$ ) has  $\gamma$  ovals if and only if there exist*

- divisors  $m_1, \dots, m_r$  of  $n$  with  $m_i > 1$  for all  $i$ ,
- divisors  $n_{11}, \dots, n_{1s_1}, \dots, n_{t1}, \dots, n_{ts_t}$  of  $d = \gcd(n, \alpha + 1)$ , with  $n_{ij} > 1$  for all  $i, j$ ,
- non-negative integers  $p_{11}, \dots, p_{1s_1}, \dots, p_{t1}, \dots, p_{ts_t}$  with  $\gcd(p_{ij}, n_{ij}) = 1$  for all  $i, j$ , and
- non-negative integers  $\varepsilon_1, \dots, \varepsilon_k, \delta_1, \dots, \delta_{m-k+1}$  with  $k \geq 1$  and  $k \geq t$ ,

such that

- (1)  $k \leq m$ ;
- (2)  $\frac{g-1}{n} = m - 1 + \sum_{i=1}^r (1 - \frac{1}{m_i}) + \frac{1}{2} \sum_{i=1}^t \sum_{j=1}^{s_i} (1 - \frac{1}{n_{ij}})$ ;
- (3)  $2m + 2r + \sum_{i=1}^t s_i > 4$ ;
- (4) the set  $\{\delta'_1, \dots, \delta'_{m-k+1}, m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  satisfies the elimination property, where  $\varepsilon'_i = n/\gcd(n, \varepsilon_i)$  and  $\delta'_j = f/\gcd(f, \delta_j)$ , with  $f = \gcd(n, \alpha - 1)$ ;
- (5)  $\varepsilon_i \cdot \frac{\alpha-1}{f} \equiv \sum_{j=1}^{s_i} \frac{d}{n_{ij}} \cdot p_{ij} \pmod{d}$  for each  $i = 1, \dots, t$ ;
- (6)  $\varepsilon_{t+i} \equiv 0 \pmod{d}$  for  $i = 1, \dots, k - t$ ;
- (7)  $\gamma = \sum_{i=1}^k \gcd(\varepsilon_i, f)$ .

**Proof.** We just point out the few specific differences with respect to the proof of Theorem 5.2.

Firstly, if there exists a symmetric Riemann surface  $X = U/\Gamma$  satisfying the hypothesis of the theorem, then the signature of the normalizer  $N_\Gamma$  of  $\Gamma$  in  $\text{PGL}(2; \mathbb{R})$  is

$$(m + 1 - k; -; [m_1, \dots, m_r]; \{(n_{11}, \dots, n_{1s_1}), \dots, (n_{t1}, \dots, n_{ts_t}), (-)^{k-t}\}),$$

with  $k < m + 1$  because the genus  $g' = m + 1 - k$  of the non-orientable quotient surface  $X/\text{Aut}^\pm X = U/N_\Gamma$  has to be positive. This proves assertion (1).

Assertions (2), (3), (5)–(7) are identical, respectively, to assertions (2), (3), (6)–(8) in Theorem 5.2, and the same happens to their proofs.

Finally, we prove assertion (4). The image of each canonical glide reflection  $d_i$  has the form  $\theta(d_i) = u^{\delta_i} \tau$ , and so its square  $\theta(d_i)^2 = u^{\delta_i(\alpha+1)}$  has order  $\delta'_i = n/\gcd(n, \delta_i(\alpha + 1)) = f/\gcd(f, \delta_i)$ , where in the last equality we have used that  $f$  and  $(\alpha + 1)/d$  are coprime. Since  $\theta$  preserves the relation  $\prod_{i=1}^{s'} d_i^2 \prod_{i=1}^r x_i \prod_{i=1}^k e_i = 1$  it follows that the set  $\{\delta'_1, \dots, \delta'_{g'}, m_1, \dots, m_r, \varepsilon'_1, \dots, \varepsilon'_k\}$  satisfies the elimination property.

The proof of the converse follows the same lines than that in Theorem 5.2, and we omit it.  $\square$

**Remarks 5.5.** (1) We observe that condition (5) in Theorem 5.2 has no analog in Theorem 5.4. This condition expresses the surjectivity of the smooth epimorphism  $\theta: N_\Gamma \rightarrow \text{Aut}^\pm X$  in the

extremal case  $k = m + 1$ , a value of  $k$  which cannot be attained for the surfaces considered in Theorem 5.4.

(2) Both in Theorems 5.2 and 5.4 we are dealing with symmetries with fixed points, and this requires the condition  $k \geq 1$ . It must be pointed out that when defining a surface kernel homomorphism  $\theta : N_\Gamma \rightarrow \text{Aut}^\pm X$ , the inequality  $k \geq 1$  plays a key role for  $\theta$  to be also surjective.

When dealing with symmetries with no fixed point, which we call *purely imaginary symmetries*, we lose the inequality  $k \geq 1$ . This gives rise to an extra condition to reflect the surjectivity, which is number (4) in the next theorem. The other conditions (1), (2) and (3) are translations of conditions (2), (3) and (4), respectively, in Theorem 5.4, taking into account that those quantities involving the parameter  $k$  merely disappear. Conditions (5), (6) and (7) in Theorem 5.4 are meaningless in this context.

**Theorem 5.6.** *Let  $g, n, m$  and  $\alpha$  be positive integers with  $g \geq 2$ ,  $n > 1$  odd and  $\alpha^2 \equiv 1 \pmod{n}$ . Then there exists a genus  $g$  symmetric Riemann surface  $X$  of type  $(n, m)$  with  $\text{Aut}^\pm X = G_n^\alpha$  and non-orientable quotient  $X/\text{Aut}^\pm X$ , whose unique symmetry (up to conjugacy within  $\text{Aut}^\pm X$ ) is purely imaginary if and only if there exist*

- divisors  $m_1, \dots, m_r$  of  $n$ , with  $m_i > 1$  for all  $i$ , and
- non-negative integers  $\delta_1, \dots, \delta_{m+1}$

such that

- (1)  $\frac{g-1}{n} = m - 1 + \sum_{i=1}^r (1 - \frac{1}{m_i})$ ;
- (2)  $m + r > 2$ ;
- (3) the set  $\{\delta'_1, \dots, \delta'_{m+1}, m_1, \dots, m_r\}$  satisfies the elimination property, where  $\delta'_i = f/\gcd(f, \delta_i)$ , with  $f = \gcd(n, \alpha - 1)$ ;
- (4)  $\text{lcm}(m_1, \dots, m_r, n/\gcd(n, \delta_i + \alpha\delta_j)) : 1 \leq i \leq j \leq m + 1 = n$ .

**Proof.** As we have already mentioned, just condition (4) requires some explanation. We keep the notations in the proof of Theorem 5.4. The group  $\text{Aut}^\pm X = \langle u, \tau \rangle$  must be generated by the elements  $\theta(d_i) = u^{\delta_i} \tau$  for  $1 \leq i \leq m + 1$  and  $\theta(x_k) = u^{\mu_k}$  for  $1 \leq k \leq r$ . This condition is equivalent to the fact that the group  $\langle u \rangle$  must be generated by the powers  $u^{\mu_k}$  and the double products  $(u^{\delta_i} \tau)(u^{\delta_j} \tau) = u^{\delta_i + \alpha\delta_j}$ . Hence, the least common multiple of the orders of these elements must be  $n$ . This is condition (4), since for  $i > j$  the element  $(u^{\delta_i} \tau)(u^{\delta_j} \tau)$  has the same order as  $(u^{\delta_j} \tau)(u^{\delta_i} \tau)$ .  $\square$

**Remark 5.7.** Theorems 5.2, 5.4 and 5.6 express the existence of a surface  $X \in \Sigma_{(g,n,m)}$  with  $\text{Aut}^\pm X = G_n^\alpha$ , with a symmetry having  $\gamma$  ovals in terms of the solvability of a finite number of arithmetical equations and congruences. They can be easily handled with the help of a computer because the variables involved are bounded, with bounds that can be expressed in terms of the data.

**Example 5.8.** Let  $n = 15$ ,  $m = 1$  and  $\alpha = 4$ . We apply the above theorems to determine all pairs  $(g, \gamma)$  such that there exists a surface  $X \in \Sigma_{(g,15,1)}^4$  with a symmetry having  $\gamma$  ovals and prescribed orientability character for the quotient  $X/\text{Aut}^\pm X$ , where

$$\text{Aut}^\pm X = G_{15}^4 = \langle u, \tau \mid u^{15} = \tau^2 = 1, \tau u \tau = u^4 \rangle.$$

Table 1

$g$	$X/\text{Aut}^\pm X$ orientable	$X/\text{Aut}^\pm X$ non-orientable
19	$\gamma = 2$	$\gamma = 1$ or 3
21	$\gamma = 2, 4$ or 6	$\gamma = 1$ or 3
23	$\gamma = 2$ or 4	$\gamma = 1$ or 3
25	$\gamma = 2$	$\gamma = 1$ or 3
27	$\gamma = 2$ or 4	$\gamma = 1$ or 3
$g$ odd, $g \geq 29$	$\gamma = 2, 4$ or 6	$\gamma = 1$ or 3

Let us briefly explain the kind of arguments involved. Note first that  $d = \gcd(15, \alpha + 1) = 5$  and  $f = \gcd(n, \alpha - 1) = 3$ . We begin with the case  $\gamma > 0$ . Since  $m = 1$  we are forced to choose  $k = 2$  in Theorem 5.2 and  $k = 1$  in Theorem 5.4. In particular,  $\gamma = \gcd(3, \varepsilon_1) + \gcd(3, \varepsilon_2) \in \{2, 4, 6\}$  in the orientable case, and  $\gamma = \gcd(3, \varepsilon_1) \in \{1, 3\}$  in the non-orientable one. In addition,  $n_{ij} = 5$  for all  $i, j$ . Let  $r_1, r_2$  and  $r_3$  be the number of divisors  $m_i$  of 15 which are equal to 3, 5 and 15, respectively, and set  $s = \sum_{i=1}^t s_i$ . With these notations, condition (2) in Theorems 5.2 and 5.4 can be rewritten as

$$g = 1 + 10r_1 + 12r_2 + 14r_3 + 6s.$$

The remaining conditions in both theorems impose restrictions on the values of the non-negative integers  $r_1, r_2, r_3$  and  $s$ . An elementary analysis of such restrictions provides Table 1.

If  $\gamma = 0$  then Theorem 5.6 yields easily that there exists a surface  $X \in \Sigma_{(g, 15, 1)}^4$  with a purely imaginary symmetry if and only if  $g$  is odd and  $g \geq 21$ , except if  $g = 23$  and  $g = 33$ .

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